Cambridge Journal of Economics 2013, **37**, 1161–1184
doi:10.1093/cje/bes056
Advance Access publication 1 March 2013

Approximate surrogate production functions

Bertram Schefold*

The Cambridge debate showed that an aggregation of capital is not possible in general. A recent investigation has found one example for reswitching and several for reverse capital deepening, but the paradoxes appear to be infrequent. The paper provides a theoretical justification of this result and shows how wage curves of input–output matrices with small non-dominant eigenvalues become quasi-linear with some numéraires. Large random systems lead to the genesis of such states. Approximate surrogate production functions then seem possible. A family of economic systems with constant capital composition allows construction of a surrogate production function.

Key Words: Capital theory, Production function, Aggregation, Random matrices
JEL classifications: C62, C67, E25

Downloaded from http://cje.oxfordjournals.org/ at Universitaetsbibliothek Johann Christian Senckenberg on September 25, 2013

1. Production functions and the paradoxes of capital

The capital theory debate, beginning with Robinson (1953–54), but really focused on Samuelson's surrogate production function (Samuelson, 1962), showed that a rigorous aggregation of capital and hence a logically stringent construction of the production function were impossible (Garegnani, 1970; Harcourt, 1972; Pasinetti, 1966; Sen, 1974). The theoretical problem turned out to differ from other aggregation problems in economics, insofar as it concerned produced means of production. Simple criteria to rule out the paradoxes failed (Gallaway and Shukla, 1976). The debate has remained open (Cohen and Harcourt, 2003) and is topical because production functions are ubiquitous in endogenous growth theory. Capital theory paradoxes have implications for intertemporal general equilibrium theory as well (Garegnani, 2003; Schefold 1997, 2005, 2008A), but these are not considered here. Anwar Shaikh (1987) extended the critique to empirical methods of estimating production functions and was answered by Solow (1987). However, the

Manuscript received 4 September 2009; final version received 8 May 2012.
Address for correspondence: FB 2, Johann Wolfgang Goethe-Universität, Frankfurt am Main, Germany; email: schefold@wiwi.uni-frankfurt.de

* Johann Wolfgang Goethe-Universität. I would like to thank Christian Bidard, Sergio Parrinello, Anwar Shaikh, Lefteris Tsoulfidis and the participants of seminars held at Duisburg, Paris, Rome, New York and Graz for valuable advice. Special thanks are due to Christian Schmidt for the empirical investigation reported in Section 4. Several referees gave precious advice, which I tried to follow. The responsibility for errors remains my own.

© The Author 2013. Published by Oxford University Press on behalf of the Cambridge Political Economy Society. All rights reserved.

opponents drew different conclusions, although they shared an analytical framework. One side still insists on its extensive use of production functions for theoretical and pragmatic reasons, the other consistently denies the legitimacy of the approach.

Each side had—and to some extent still has—different economic paradigms in mind, which reflect different visions of the working of capitalism. The paradigms are rooted in the neoclassical schools, on the one hand, and in Keynesian and Marxian traditions, on the other (Schefold, 1989 [1971], part III; Schefold, 2004). But the underlying problem of capital theory cuts across these boundaries, e.g. the fact that the value of the physical capital employed by an economy depends on distribution also renders Keynesian theories of growth more difficult (Schefold, 1997, p. 276) and it is connected with the transformation problem. Without denying the importance of these other aspects, we here focus on the implications for the production function and mention broader implications only cursorily. This focus is possible because both sides in the debate around 1960–80 were content to use essentially the same highly abstract analytical framework that we describe in this section. Theoretical advances made since and new empirical insights now make it necessary to modify the framework. Not all aspects of this theoretical departure are dealt with in this present paper, for reasons of space. For instance, my reasons for remaining closer to Joan Robinson's account of technical choice as given in the middle of her career and for not fully accepting her later self-criticism for her earlier use of 'books of blueprints' can only be discussed elsewhere (Schefold, 2013).

Without going into the ramifications, we tackle the concrete problem. Recently, an empirical inquiry (Han and Schefold, 2006) showed that empirical examples of reverse capital deepening (see below in this section) exist, but are not frequent. If such results prove to be robust (Schefold, 2012), a theoretical explanation, convincing for both sides in the debate, must be sought. At any rate, it is our purpose to take steps in this direction.[1]

The name of the surrogate production function already suggested that its originator, Samuelson (1962), had something less than perfect in mind. As is well known, a sufficient condition for a production function to exist in a closed economy with heterogeneous capital goods and labour is that all these capital goods are of one kind, and many believe this condition also to be necessary, but this is a mistake, as we shall show. It would be desirable to identify conditions that are both necessary and sufficient for existence, but such conditions can be formulated only in a specific theoretical framework and would be cumbersome, as we shall also see. The point is to identify sufficient conditions that are economically meaningful and to argue about their realism.

An economy, for which a production function with constant returns to scale exists, will itself have to exhibit constant returns to scale. The analysis must take place in a long-term context, since the equality of marginal products and factor prices without quasi-rents for firms obtains only in the long term; necessarily, prices are long-term cost-of-production *normal* prices, including normal profits at a prevailing rate profit, equal to the rate of interest, if we abstract from special risks and entrepreneurial effort. Hence, the assumptions made for the construction of the surrogate production function are constant returns to scale, single-product industries and labour of uniform quality. There is no reason to generalise at this stage, since the introduction of heterogeneous labour, of fixed capital and joint production and of variable returns to scale does not render the existence of the surrogate production function more likely. No

[1] For a first incomplete enunciation of some of the results of this paper (Sections 1–4 only) see Schefold (2008B).

special form of the production function will be postulated.[2] The assumption of perfect competition should be retained.[3]

We thus assume a finite number of methods of production, available for the production in each industry in the form of a book of blueprints. Competition will ensure that, at any given rate of profit, a certain combination of methods will be chosen, one in each industry, such that positive normal prices and a positive wage rate result, expressed in terms of a numéraire. The wage rate can then be drawn as a function of the rate of profit for this combination of methods between a rate of profit equal to zero and a maximum rate of profit, and the 'individual' wage curve for this technique will be monotonically falling (see Han and Schefold (2006) for a more detailed description). If the choice of technique is repeated at each rate of profit, starting from zero, different individual wage curves will appear on the envelope of all possible wage curves and the envelope will also be monotonically falling. The choice of technique is '*piecemeal*' in that only one individual wage curve will be optimal in entire intervals, except at a finite number of switchpoints where generically only two wage curves intersect and where a change of method generically takes place only in one industry, so that the two wage curves to the left and to the right of the switchpoint will have all other methods in all other industries in common.[4] The intensity of capital and output per head change discontinuously at the switchpoints (they can be represented geometrically for a given individual wage curve, $w(r)$, if the numéraire consists of the vector of output per head in the stationary state): output per head equals $w(0)$ and capital per head $k = (w(0) - w(r))/r$ (see Figure 1).

If many individual wage curves appear successively on the envelope, this envelope may be replaced by a smooth approximation and each point on this modified envelope can be thought to represent one individual technique, represented by an individual wage curve. The surrogate production function, then, is defined by taking the tangent to this modified envelope (supposed to be convex to the origin): the slope of the tangent is equal to capital per head and the intersection of the tangent with the abscissa is equal to output per head, as in Figure 1. One thus obtains capital per head $k(r)$ and output per head $y(r)$ as functions of the rate of profit, and one can show from this parameter representation that a production function $y = f(k)$ must exist (Schefold, 1989, pp. 297–8). If, and only if, the individual wage curves are linear, the construction is rigorous in that output per head and capital per head of techniques individually employed will be equal to those that we have just defined, and the paradoxes of capital theory (which will be discussed presently) will then be absent.

However, the critique of the surrogate production functions starts from the observation that individual wage curves will in general not be linear and the envelope will not be necessarily convex to the origin; envelope $\hat{w}(r)$ in Figure 1 provides an example.

[2] The likelihood of the existence of CES production functions is discussed in Schefold (2006).

[3] A set-theoretical description of technological alternatives does not eliminate the possibility of paradoxes of capital theory, as long as strict convexity is not postulated, and strict convexity is an extremely problematic assumption (see Schefold, 1976).

[4] We are here not concerned with changes of technique that require the replacement of several processes at once for technical reasons, such as the introduction of a new good, produced by a new machine, as such additional complications may be dealt with later and in combination with the introduction of fixed capital. The reader should also note that we are here not concerned either with two-sector models, with a capital-goods sector and a consumption-goods sector, in which capital is regarded as a different good at each level of the rate of profit. Capital goods here are heterogeneously produced means of production and each has its value in use, which is independent of prices and the rate of profit.

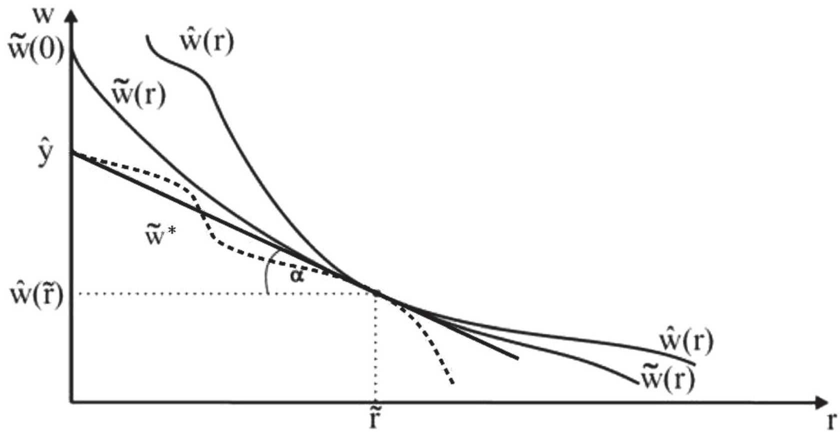

Fig. 1. *Individual wage curve $\tilde{w}(r)$ touches the envelope of wage curves $\hat{w}(r)$ at $\tilde{r}$; declination then is $\tilde{w}(0)-\hat{y}$. If the shape of $\tilde{w}$ happened to be given by $\tilde{w}^*$, declination would vanish. The surrogate production function yields output per head $\hat{y}=\hat{w}(\tilde{r})+rtg\alpha$, $tg\alpha=-\hat{w}'(\tilde{r})=K/L$, since $tg\alpha=(\hat{y}-\hat{w})/\tilde{r}$, but actual output per head equals $\tilde{w}(0)$ in the stationary state.*

Output per head at $\tilde{r}$ is given by $\tilde{w}(0)$, where $\tilde{w}(r)$ is the individual wage curve tangent to $\hat{w}(r)$ at $\tilde{r}$. With non-linear wage curves, there is likely to be a divergence between output per head and capital per head in the various industries, relative to their common value implicit in the definition of a surrogate production function. The overall divergence is termed *declination* in Figure 1: output per head would be $\hat{y}$ and $k=tg\alpha$ if the individual wage curve $\tilde{w}(r)$ was linear, but since this is not the case, there is the declination $\tilde{w}(0)-\hat{y}$. Output per head equals $\hat{y}$ according to the definition of the surrogate production function, but actual output per head really is $\tilde{w}(0)$ if $r = 0$.

Declination entails an inaccuracy of the procedure of aggregation. The paradoxes of capital concern the change of techniques, engendered by the change of distribution and are visible in the piecemeal change of wage curves on the envelope. The phenomenon that has attracted most attention is that of reswitching and reverse capital deepening: there may be switchpoints on the original envelope such that the intensity of capital does not fall with the rate of profit (*reverse capital deepening*) and the individual wage curve may have appeared on the envelope already at a lower rate of profit (*reswitching*). It is also possible that capital per head rises with the rate of profit in the industry where the switch of methods of production takes place (*reverse substitution of labour*) and, surprisingly, reverse capital deepening (the perverse change of aggregate capital per head) and reverse substitution of labour (a perverse change of capital per head at the industry level) need not go together[5] in systems with more than two industries (Han and Schefold, 2006). *Returns of processes* seem to be frequent: a process that is used in

[5] The main case in which this *paradox of paradoxes* occurs is given, if reverse capital deepening at r_3 is associated with three switchpoints r_1, r_2, r_3 between two wage curves $w^\star$, $w^{\star\star}$, with $-1<r_1<0<r_2<r_3<Min(R^*,R^{**})$; $R^\star$, $R^{\star\star}$ maximum rates of profit; with the switch at r_3 on the envelope, with $w^\star$ above $w^{\star\star}$ at $r = 0$ and for $r > r_3$, and with the intersection at r_2 dominated by some third wage curve w^+, $R^+ < r_3$ (the reader is advised to draw the diagram). As r rises from $r_3 - \varepsilon$ to $r_3 + \varepsilon$, $\varepsilon > 0$ in a stationary state, the aggregate intensity of capital rises paradoxically, since $w^\star(0) > w^{\star\star}(0)$. But the constellation also implies that, as the switch takes place in one sector, say 1, l_1 increases. For, with **d** as numéraire, we have $w^*(-1)<w^{**}(-1)$

one industry in one interval of the rate of profit is used again in another interval, but not in between. This is a generalisation of reverse capital deepening. It can be shown to imply large changes of relative prices and capital values and it demonstrates that processes cannot be classed as being inherently more or less capital intensive, prior to their use in specific systems and at specific levels of distribution.

Some thought (e.g. myself; Schefold, 1989 [1971], p. 298) that reverse capital deepening might be about just as likely (frequent) as 'normal' switches and that one would encounter 'many' individual wage curves succeeding each other on the envelope in a piecemeal fashion (it was conceded that 'reswitching' might be 'rather unlikely' in Schefold (1997, p. 480)). A different picture emerges in Han and Schefold (2006), where it is assumed that techniques used in the past, as represented in corresponding input–output tables, could be used again and that similarly the technique used in another country could be used at home. Comparing only two input–output tables in this manner results in a multitude of wage curves, since two methods (the foreign method or that of the past) are available as alternatives to the actual method employed in each industry so that 2^n alternative systems result, if both input–output tables are composed of n sectors.

Han and Schefold (2006) analysed envelopes derived from nearly 500 pairs of input–output tables for economies different in space or time (32 tables with 36 sectors). It was not possible to compute all the 2^{36} wage curves for each of $(32 \times 31)/2 = 496$ pairs, but the envelopes were obtained by means of linear programming. Contrary to our expectations, reverse capital deepening and reverse substitution of labour are obtained only in a little less than 4% of all switchpoints on the envelopes. Technical change is confirmed as piecemeal, but, also surprisingly, only about 10 wage curves out of $2^{36} = 68\,719\,476\,736$ appear on average on each envelope. (This number will increase if more than two—say m—input–output tables of n sectors are combined to define a book of blueprints.)

Joan Robinson used to tell me that if one technique was really better than another, their wage curves would not cross at all—I replied that the stylised facts of growth theory (constant capital–output ratio and weak dependence of the capital–labour ratio of a given technique on the rate of profit) implied near-linear wage curves turning around the maximum rate of profit (Schefold, 1997, p. 277 (paper first published 1979)); hence, with perturbed techniques, one would have to expect some switchpoints near the maximum rate of profit. At the other extreme, neoclassical theory and Sraffa share the expectation that, as one moves down the envelope, there will be a 'rapid succession of switches' (Sraffa, 1960, p. 85).

To look at the effect of all '*combinations*' of methods on the envelope was the starting point of the analysis by Han and Schefold (2006). Similar empirical investigations would be welcome to confirm or question our results. There are considerable methodological problems; they are discussed in the paper itself. Meanwhile, it has become

with $1 = \mathbf{dp}^*(-1) = \mathbf{dp}^{**}(-1) = w^*(-1)\mathbf{dl}^* = w^{**}(-1)\mathbf{dl}^{**}$, hence $\mathbf{dl}^* > \mathbf{dl}^{**}$ and, since $l_i^* = l_i^{**}$; $i = 2,\dots,n$; $l_1^* > l_1^{**}$. By definition of a switchpoint $(1+r)\mathbf{a}_1{}^*\mathbf{p}^* + w^* l_1^* = (1+r)\mathbf{a}_1{}^{**}\mathbf{p}^{**} + w^{**} l_1^{**}$ and $w^\star = w^{\star\star}$ at r_3. We conclude that raising r from $r_3 - \varepsilon$ to $r_3 + \varepsilon$ leads to a fall in the intensity of capital in the sector, where the switch in the method of production occurs, while the intensity of capital in the aggregate rises. Reverse capital deepening, a rise of the intensity of capital as the rate of profit rises, is curious enough, but it is even more curious that this can happen while the intensity of capital falls, as usually expected, in the sectors where the change of technique takes place. The phenomenon should not be confused with a Wicksell effect.

necessary to reflect on this peculiar outcome theoretically. The critics of neoclassical theory can point out that, for the first time, an empirical case of reswitching and many of reverse capital deepening have been found. But the frequency is not sufficient to destroy neoclassical hopes that the production function might survive as an approximation, similar perhaps not to the more rigorous laws of physics but to the empirical generalisations, supported by some theoretical considerations, which one finds in biology. The discussion then moves on a plane lower than that of the critique of pure theory in which approximations are not permitted. There must be theories also for approximations in the measurement of capital. It once was appropriate to confront the measurement 'in which the statisticians were mainly interested' (only 'approximate') and 'theoretical measures' that 'required absolute precision' and corresponded to 'pure definitions of capital', as 'required' by the theories (statement by Piero Sraffa of 1958, as quoted by Sen (1974, p. 331)). Sraffa was concerned with the 'theoretical measures', but we now want to create a theory for the approximations.

If the individual wage curves were linear, the envelope would become convex to the origin, declination would vanish and the intensity of capital would fall with any increase of the rate of profit. This would be a sufficient condition for the existence of a surrogate production function, but it would not be strictly necessary, since, even if the individual wage curve is not straight, declination might vanish by accident, as in the case of the alternative shape $\tilde{w}^*$ of $\tilde{w}$ in Figure 1. But this coincidence could hardly be expected to occur for each individual wage curve at different levels of the rate of profit. In essence, therefore, the linearity of individual wage curves is a necessary condition for the existence of a surrogate production function.

But the consideration shows that not strict but only approximate linearity is needed if we are interested in a theory for the approximations. The open question, thus, is whether the surrogate production function can be defined under assumptions that are sufficiently general to take the relevant aspects of real modern economies into account and sufficiently specific to show that declination is so small and the paradoxes of the capital are so rare that they can be ignored. This construct—if it exists—could be called an '*approximate surrogate production function*'.

2. Foundations of the approximation

The original surrogate production function had linear wage curves and strictly linear wage curves imply that prices are equal to *labour values* (unless the numéraire is very special). Prices and values can differ substantially, as Ian Steedman and Judith Tomkins (1998) assert. It would not only be ironic to fall back on a primitive form of the labour theory of value (Marx had prices of production as transformed labour values), but there is also a specific inconsistency implied by the assumption of prices equal to values: it can be shown that two techniques with linear wage curves, due to uniform organic compositions of capital, cannot coexist at a switchpoint; the switch would violate the principle of combination. For if their linear wage curves cross, a combination of the methods of the techniques will exist, with a wage curve dominating this point of intersection (Salvadori and Steedman, 1988). The reason is that technical change on the envelope must be piecemeal. If we have a wage curve for a technique with uniform composition of capital on the envelope, more than one method must change in order to get to another technique that is also characterised by a uniform composition of capital.

A linear wage curve also results if the basket of goods defining the numéraire happens to be equal to Sraffa's *standard commodity*. The deeper reason why wage curves otherwise are not exactly straight derives from a property of the so-called '*regular*' Sraffa systems introduced by Schefold (1989 [1971]): a system (**A**, **l**) is regular if the eigenvalues of **A** are semi-simple and if **l** is not orthogonal to any of the left-hand eigenvectors of **A**. This property is generic and equivalent to the linear independence of the vectors **l**, **Al**,..., $\mathbf{A}^{n-1}\mathbf{l}$. The point here is that it is also equivalent to the linear independence of the price vector $\mathbf{p}(r)$, taken at n different rates of profit, i.e. to the linear independence of $\mathbf{p}(r_1),\ldots,\mathbf{p}(r_n)$; $r_1 <\ldots< r_n$. The implied movement of relative prices entails a not vanishing curvature of $w(r)$, unless the numéraire happens to be an eigenvector.

The two constellations mentioned above, which lead to linear wage curves, both concern the eigenvectors of the input matrix. If the labour theory of value holds and relative prices are constant, they must be equal to the relative prices formally obtained at a rate of profit equal to −1. They will then be equal to relative direct labour inputs. Hence, the labour vector must be the Frobenius eigenvector of the input matrix if the labour theory of value holds. The standard commodity, on the other hand, is known to be the dual positive eigenvector. In the former case, the linear wage curve is possible because the system is not regular; in the second, because the numéraire is an eigenvector, which also implies an irregularity according to the extended definition in Schefold (1997, p. 116). Schefold (1989 [1971]) further considered the other eigenvalues of the input matrix. A transformation, which will be used again here, showed that relative prices as functions of the rate of profit took a very simple form, related to the properties of Sraffa's standard system, if the eigenvalues other than the Frobenius eigenvalue were zero. Thirty years later, Christian Bidard proved a hypothesis by Bródy and showed in a seminal paper together with Tom Schatteman (Bidard and Schatteman, 2001) that the eigenvalues other than the dominant eigenvalue will tend to zero for larger and larger random matrices, and their result has been generalised and proved independently by mathematicians since (see Appendix for the mathematical definition of random matrices and for the theorems referred to). On this basis, one can show (see Section 3) that large *'random' systems* will exhibit hyperbolic wage curves, i.e. wage curves of uniform curvature without 'wiggles' (the 'wiggles' are responsible for the 'paradox of paradoxes' that occurs if wage curves have more than two switchpoints, as shown in note 5 above).

We thus have three properties on which the construction of approximate surrogate production functions might perhaps be based, because they lead to nearly linear wage curves and so reduce the risk of paradoxes and large declinations: they would be based on systems with prices not differing much from labour values, with numéraire vectors not differing much from the standard commodity and with matrices having small eigenvalues (except for the dominant one). We shall see in Section 4 that the two former conditions can be relaxed dramatically, provided the latter holds.

However, there are *three additional supporting properties*. One can observe that the magnitudes on which the paradoxes of capital depend are, from the formal mathematical point of view (economically, the spectrum of techniques is assumed to be discrete), locally *continuous functions* of elements of the input matrix, of the labour vector and of the numéraire, so that each single small change of methods of production in different industries, such as occurs at a switchpoint, can only exert a small effect on the aggregates. If the system is large and the changes are many, rare paradoxical changes will, as it were, disappear in the noise of frequent transitions (the empirical results in Han and Schefold (2006) had this character[6]). The argument fails if the paradoxes are frequent. That the paradoxes are rare would have to be confirmed by means of further empirical studies and will here be supported theoretically, primarily by means of the first three arguments.

The fifth argument concerns declination only and is discussed in Schefold (2006): one can prove that declination will diminish if a positive rate of growth g is introduced and declination disappears in the *golden rule* case $r = g$. Reswitching, in contrast, exists also in the golden rule case and independently of the choice of the numéraire: a technique that had been in use at low rates of profit reappears at high rates. But capital per head falls at both switchpoints, since declination disappears with $r = g$. If the first switchpoint is dominated by a third technique, capital reversing will thus not be observed on the envelope and two effects (the change of the quantity of capital, induced by the change of the rate of growth, and the change in the value of capital, induced by the change in the rate of profit) compensate each other. This golden rule case is only of theoretical interest, however, since the important applications of the production function concern problems of employment and distribution that typically occur at low or zero rates of growth (in particular, there is unemployment in a stagnant economy and the question is whether lowering wages and raising the rate of profit will induce a choice of technique that eliminates unemployment). If growth is negligible, declination obviously increases with the curvature of the individual wage curve that appears on the envelope. If this curvature does not change signs (no 'wiggles'), declination increases with the level of the rate of profit. If there are 'wiggles', the paradoxes become possible.

The last argument is statistical and concerns randomness in a broader sense than that of random matrices—e.g. the labour vector could also be random. Randomness leads not only to small eigenvalues for large random matrices (as affirmed above and shown in the Appendix), but also lends stability to all aggregates—capital, income and its components, as was observed by Marx. He based his assertion that total profits could be represented as a redistribution of surplus value partly on an erroneous algebraic 'transformation' of values into prices and partly on the hypothesis, made a priori but not implausible, that the statistical deviations of prices from values were random and would roughly tend to cancel on average. Here we can state in a like way that changes of distribution may have large effects on the relative prices of certain capital goods, but only a smaller effect on the aggregate price of all capital goods and on the components of income. We thus get two opposing tendencies. As we consider larger systems with an increasing number of commodities n, the degree of the polynomials in r describing the movement of relative prices increases—this is the algebraic argument, but the aggregates become more stable on account of randomness. Which tendency prevails?

The reader should note that we are here not talking about the uncertainty of the measurement of individual coefficients of the system. The error involved in the measurement of individual elements of the matrix can be quite different according to the industry and the input concerned and must be reflected in distributions of the likely magnitude of the input that are specific to this element. The uncertainty about the

[6] See Table 2 in Han and Schefold (2006), where reverse capital deepening—where it occurs—results in a change of the value of capital of the order of magnitude of 1%.

inputs of random matrices, in contrast, comprises an uncertainty regarding the methods of production; only a specific mean is assumed to be given for the distribution of coefficients in each industry. The deterministic counterparts of the systems so defined are irregular. Schefold had shown in 1971 (see Schefold 1989 [1971]) that the neoclassical construction is based on irregular systems and that irregular systems are of measure zero in the set of all systems and thus not generic. Surrogate production functions are therefore definitely not rigorous if the systems are regular. But it now seems possible that large random systems are not far from an 'irregular' state and irregular systems, though not generic, might turn out to represent valid approximations to reality. We might go further and say that they play a role similar to that played by *attractors* in chaos theory, since the non-dominant eigenvalues of random matrices converge to zero with increasing n, as we shall see in the Appendix, and n increases with the growth of the economy.

We start from the first three arguments in this section, which concern the forms of the individual wage curves and therefore both paradoxes and declination. The randomness of matrices will be considered in Section 3, a further element of randomness will be introduced in Section 4 and continuity will be invoked in Section 5, but we shall not use the golden rule assumption to eliminate declination. My investigations have led me to the conviction that no single one of the first three properties can serve to justify the construction of an approximate surrogate production function. Whether combinations of them (or of all six effects) can do that is again our open question in a more developed form.

In a preliminary attempt to solve it (Schefold, 2008B), I proposed to discuss 'families' of wage curves defined by some common properties of the techniques involved. The families were called 'closed' if combinations of two techniques and their wage curves lead to a combined technique and wage curve that still belonged to the same family. Three such 'families' were discussed. One, based on '*circular*' systems as extensions of '*Austrian*' (Schefold, 1999) models,[7] was used to show that wage curves with extreme curvature are possible. The second, on the contrary, is taken up again here, using a more general notion of randomness than in Schefold (2008B), in order to demonstrate how near linearity may result. Since this family is associated with random 'large' input–output systems, the result justifies the use[8] of approximate surrogate production functions to the extent that real systems are random. A third family will eventually be found, the deterministic counterpart of the second, that exhibits strictly linear wage curves and thus permits the construction of a rigorous surrogate production function, even though more than one commodity is produced.

3. Systems with small non-dominant eigenvalues

The techniques can be represented by Sraffa systems (Sraffa, 1960) of the usual form:

$$(1 + \mathrm{r})\mathbf{A}\mathbf{p} + \mathrm{w}\mathbf{l} = \mathbf{p},$$

[7] Circular systems focus attention on the contrast between the ease with which examples of reswitching of the wine-and-oak-chest type could be constructed (Sraffa, 1960) and the difficulty of finding reswitching in interdependent systems. This contrast was at the origin of the false hypothesis advanced by Levhari (1965) that reswitching would not occur in an interdependent basic system (the possibility of a continuous transition from non-basic to basic systems was noted only afterwards by Levhari and Samuelson, 1966, p. 519).

[8] Perhaps it would be better to say 'justifies the *construction* of an approximate surrogate production function', as the use to which the construction might be put would have to be discussed in a much broader context and would require more space than is available in this paper.

where $\mathbf{A} = (a_{ij})$ is the input matrix, $\mathbf{l} = (l_i)$ is the (positive) labour vector (column), $\mathbf{p} = (p_i)$ is the (row) vector of prices; $i, j = 1,\ldots,n$; w is the wage rate, r is the rate of profit and $\mathbf{d} = (d_1,\ldots,d_n)$ is the numéraire vector (row); prices are normalised so that $\mathbf{dp} = 1$ for all r. The systems are assumed to be semi-positive, basic (indecomposable) and productive. Productivity can be ensured by assuming that there is a surplus with $\mathbf{eA} \leq \mathbf{e}$ ($\mathbf{e}$ is the summation vector). The prices expressed in this numéraire and the wage rate will then be positive for $0 \leq r \leq R$,[9] with R being the maximum rate of profit.

We now assume that the non-dominant eigenvalues of the input–output systems are small. As Bidard and Schatteman (2001) have shown, in the article already quoted, the non-dominant eigenvalues of so-called *random matrices* (with a random distribution of positive coefficients) all tend to zero as the number of sectors increases. For an intuitive argument to explain this surprising result, followed by a formal statement of the underlying theorem in a more advanced mathematical form, see the Appendix.

Of course, random matrices are not the only matrices with small non-dominant eigenvalues. Matrices with non-dominant eigenvalues strictly equal to zero will be discussed in Section 5. It is clear that the elements of actual input–output tables are not strictly random: they are not independent, in that if, for example, a_{ij} is a chemical used in the production of a pharmaceutical product i, the quantity a_{ik} may denote another chemical required in a precise amount. For the time being, we assume an identical distribution in each row out of a priori ignorance.

We start with $\mathbf{A} \geq 0$ basic, with eigenvalues written as $(1+R_i)^{-1}$; $i = 1,\ldots,n$; where $R_2,\ldots,R_n$ are different 'large' maximum rates of profit (except for the 'true' maximum rate of profit R_1, which corresponds to the Frobenius eigenvalue).[10] We have $(1 + R_i)\mathbf{q}_i\mathbf{A} = \mathbf{q}_i$, $\mathbf{l} > 0$, $\mathbf{d} \geq 0$. With any of the associated left-hand eigenvectors we get (proof by inversion of the matrix):

$$\mathbf{q}_i\left(\mathbf{I}-(1+r)\mathbf{A}\right)^{-1} = \frac{1+R_1}{R_i - r}\mathbf{q}_i.$$

This is a generalisation of Sraffa's standard system, where $\mathbf{d} = \mathbf{q}_1 = \mathbf{q}(\mathbf{I} - \mathbf{A})$, $R_1 = R$ is the maximum rate of profit, with normalisation $\mathbf{ql} = 1$, $\mathbf{el} = 1$; this $\mathbf{d}$, taken as the numéraire, yields Sraffa's familiar linear wage curve in terms of standard prices $\bar{\mathbf{p}}$:

$$1 = \mathbf{q}(\mathbf{I}-\mathbf{A})\bar{\mathbf{p}} = r\mathbf{q}\mathrm{A}\bar{\mathbf{p}} + \bar{w}\mathbf{ql} = (r/R)\mathbf{q}(\mathbf{I}-\mathbf{A})\bar{\mathbf{p}} + \bar{w}\mathbf{ql} = (r/R) + \bar{w}.$$

One thus has the wage curve in terms of the standard commodity:

$$\bar{w} = 1 - \frac{r}{R}.$$

We now choose any arbitrary numéraire $\mathbf{d} > 0$ and represent the numéraire as a linear combination of the eigenvectors (rows): $\mathbf{d} = \lambda_1\mathbf{q}_1 + \ldots + \lambda_n\mathbf{q}_n$. Using also the dual right-hand eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (columns), we represent the labour vectors as $\mathbf{l} = \gamma_1\mathbf{x}_1 + \ldots + \gamma_n\mathbf{x}_n$.

We thus obtain a simplified formula for the inverse of the wage rate:

[9] The wage is assumed to be paid *ex post*, as in Sraffa (1960), since the same assumption is made when the production function is used to show the equality of the marginal product of labour and the real wage rate.

[10] We make the generic assumption that the eigenvalues are different (Gantmacher 1966 [1959], Gröbner 1966).

$$\frac{1}{w} = \mathbf{d}\left(\mathbf{I}-(1+r)\mathbf{A}\right)^{-1}\mathbf{1} = \Sigma\lambda_i \mathbf{q}_i\left(\mathbf{I}-(1+r)\mathbf{A}\right)^{-1}\mathbf{1} = \Sigma\lambda_i \frac{1+R_i}{R_i - r}\mathbf{q}_i\mathbf{1}.$$

We are now interested in a family of wage curves for which the absolute values of $R_2,\ldots,R_n$ are large enough so that each r/R_i for $0 \leq r < R$ and $1/R_i$ can be ignored. This will be the case, in particular, for random matrices, as $1/(1 + R_i)$ will then tend to zero almost surely with $c(p)/\sqrt{n}$, p probability, $0 \leq p < 1$, c constant, according to Goldberg *et al.* (2000, p. 150). This yields an approximate wage curve $\tilde{w}$ ($R = R_1; (1+R_i)/(R_i - r) \to 1; i = 2,\ldots,n$):

$$\frac{1}{\tilde{w}} = \left[\frac{1+R}{R-r}\lambda_1\mathbf{q}_1 + \lambda_2\mathbf{q}_2 + \ldots + \lambda_n\mathbf{q}_n\right]\left[\gamma_1\mathbf{x}_1 + \ldots + \gamma_n\mathbf{x}_n\right].$$

Left-hand and right-hand eigenvectors belonging to different eigenvalues are orthogonal; hence, $\mathbf{q}_i\mathbf{x}_j = 0$ for $i \neq j$. We introduce a *strong normalisation* by choosing the eigenvectors such that $\lambda_i = \gamma_i = 1$ for all i. As a result, the approximate wage curve may be written as:

$$1/\tilde{w} = \left[(1+R)/(R-r)\right]\mathbf{q}_1\mathbf{x}_1 + \mathbf{q}_2\mathbf{x}_2 + \ldots + \mathbf{q}_n\mathbf{x}_n$$

or

$$\tilde{w} = \frac{R-r}{(1+R)\mathbf{q}_1\mathbf{x}_1 + (\mathbf{q}_2\mathbf{x}_2 + \ldots + \mathbf{q}_n\mathbf{x}_n)(R-r)}$$

with $\mathbf{d} = \mathbf{q}_1 + \ldots + \mathbf{q}_n$, $\mathbf{1} = \mathbf{x}_1 + \ldots + \mathbf{x}_n$. In abbreviated form:

$$\tilde{w}(r) = \frac{R-r}{a-zr}.$$

Clearly, $\mathbf{q}_1\mathbf{x}_1 > 0$ and $\tilde{w}$ and $R-r$ are real for real values of r, hence $z = \mathbf{q}_2\mathbf{x}_2 + \ldots + \mathbf{q}_n\mathbf{x}_n$ must be real and $a = \mathbf{q}_1\mathbf{x}_1 + R(\mathbf{q}_1\mathbf{x}_1 + \ldots + \mathbf{q}_n\mathbf{x}_n) > 0$ from $\tilde{w}(0) > 0$, $R > 0$. Two cases result, represented by two hyperbolas: one concave, one convex (for diagrams see Schefold, 2008B).

4. First conclusions and discussion of the main assumption

The actual wage curve $w(r)$ will be very close to the hyperbola $\tilde{w}(r)$ and may cross it several times. And it is clear that the hyperbola given by $\tilde{w}$ will approximate the linear wage curve of the standard wage $\bar{w}$ the better, the closer z is to zero, for the asymptotes will then move to infinity and the wage curve $\tilde{w}$ will become linear. The case favourable for the construction of the surrogate production function and for neoclassical theory is obtained with $z < 0$, for the hyperbola will then be convex; it will be relatively straight if $|z|$ is small. A positive z implies $0 < z < a/R$, since the wage curve cannot diverge to infinity for $0 \leq r \leq R$.

We thus identify two properties of the systems, which lead together to almost linear wage curves:

(1) If the non-dominant eigenvalues of the matrix are small enough, a simple hyperbolic form of the wage curve results; it is, as it were, very smooth. The wage curve, which in general is given by the ratio of two polynomials in r, of degree $n - 1$ and n, respectively, reduces here to a ratio of two polynomials of the first degree. *We conclude that any wage curve that is more complicated than a simple hyperbola owes these complications to non-dominant eigenvalues that are not equal to zero.*

(2) If we have a hyperbola and want to obtain a nearly linear wage curve, it is important that z is close to zero so that the hyperbola is 'stretched'. For this it is sufficient that z is *small*.

We can define a row vector $\mathbf{m}$ of the deviations of the numéraire vector $\mathbf{d}$ from the strongly normalised standard vector $\mathbf{q}_1$, which is the left-hand side eigenvector of $\mathbf{A}$, with

$$\mathbf{d} - \mathbf{q}_1 = \mathbf{q}_2 + \ldots + \mathbf{q}_n = \mathbf{m},$$

and similarly column vector $\mathbf{v}$ denotes the deviations of the actual labour vector from the strongly normalised right-hand side Frobenius eigenvector of $\mathbf{A}$:

$$\mathbf{1} - \mathbf{x}_1 = \mathbf{x}_2 + \ldots + \mathbf{x}_n = \mathbf{v};$$

$\mathbf{m}$ and $\mathbf{v}$ are real. Now $z = 0$ and the wage curve $\tilde{w}$ is linear if $\mathbf{m} = 0$ (standard commodity case) or if $\mathbf{v} = 0$ (labour theory of value case), but z is zero also if, using $\mathbf{q}_i\mathbf{x}_j = 0$ for $i \neq j$, $\mathbf{mv} = \mathbf{q}_2\mathbf{x}_2 + \ldots + \mathbf{q}_n\mathbf{x}_n = 0$. At first, one cannot think of any reason why this orthogonality should occur—to assume it seems to result in overkill: the wage curve could be strictly linear without either $\mathbf{m}$ or $\mathbf{v}$ being strictly equal to zero. But it is consistent with our approach now to consider the components of $\mathbf{m}$ and $\mathbf{v}$ as *independent random variables* with *small*[11] means. This leads to an important new result: because of the assumed independence of $\mathbf{m}$ and $\mathbf{v}$, we get, with $\bar{m}$, $\bar{v}$ as means of the components $m_1, \ldots, m_n$; $v_1, \ldots, v_n$:

$$\begin{aligned} 0 = \operatorname{cov}(\mathbf{m}, \mathbf{v}) &= (1/n)\sum_{i=1}^{n}(m_i - \bar{m})(v_i - \bar{v}) \\ &= \mathbf{mv}/n - (m_1 + \ldots + m_n)\bar{v}/n - \bar{m}(v_1 + \ldots + v_n)/n + n\bar{m}\bar{v}/n \\ &= \mathbf{mv}/n - \bar{m}\bar{v}, \end{aligned}$$

hence

$$z = \mathbf{mv} = n\bar{m}\bar{v}$$

is small of the second order if $\bar{m}$, $\bar{v}$ are—given n—small of the first order.

There is room for different assumptions as to how $\bar{m}$ and $\bar{v}$ might vary with n. It seems more plausible that $\bar{v}$ will tend to zero than $\bar{m}$, since the labour vector may be random for reasons at least as good as those adduced for the randomness of $\mathbf{A}$, while, even if $\mathbf{m} = 0$ for one of the techniques in the book of blueprints, $\bar{m}$ may

[11] Example with $n = 2$: $\mathbf{d} = (1,1)$, $\mathbf{q}_1 = (0.9, 1.2)$, $\mathbf{q}_2 = (0.1, -0.2)$, $\mathbf{1}^{\mathrm{T}} = (1,1)$, $\mathbf{x}_1^{\mathrm{T}} = (1.1, 0.9)$. $\mathbf{x}_2^{\mathrm{T}} = (-0.1, 0.1)$ yields $\mathbf{q}_1\mathbf{x}_1 = (0.9 \times 1.1 + 1.2 \times 0.9) = 2.07$, $\mathbf{q}_2\mathbf{x}_2 = (-0.1 \times 0.1 - 0.2 \times 0.1) = -0.03$.

tend to become larger as one moves away from that technique to other techniques on the envelope of the wage curves. We call this increase of $\bar{m}$ the *effect of the numéraire drift*. The possibility follows from neoclassical theory itself. If, for example, techniques become more mechanised as the rate of profit falls, the left-hand eigenvectors of the input matrices will move away from **d** and the deviations increase, not to infinity but possibly resulting in a mean deviation $\bar{m}$ significantly different from zero. If the correlation between **m** and **v** continues to be negligible, the individual wage curves will continue to be near linear, if, for each technique, $\bar{v}$ is small enough to compensate for an $\bar{m}$ that will be appreciably different from zero.

We thus first get a simple mathematical confirmation of what we affirmed above: the properties of randomness of **A**, of prices being near values, of the numéraire being close to the standard commodity, reinforce each other. For if the matrix is not random, 'wiggles' may remain, leading to the paradoxes. If neither of the two other properties is fulfilled, the matrix being random, the hyperbola can deviate from linearity to an appreciable extent, depending on $\bar{m}$ and $\bar{v}$, as seems to be the case for empirical wage curves, derived from input–output systems, if $\bar{v}$ is not very small and if numéraires far from standard proportions are chosen.

However, it is fairly obvious, after all, that wage curves will be nearly linear, if prices remain close to values and/or if the numéraire is close to the standard commodity. Note now, second, that we have proved much more than these 'classical' statements. We have shown that, if **m** and **v** are uncorrelated for any given technique with zero non-dominant eigenvalues, a linear wage curve will be obtained, if either $\bar{m}=0$ or $\bar{v}=0$. The variances of **m** and **v** do not matter and the statement could be extended: if $\bar{v}$ is small enough, the wage curve will be nearly linear. A possible application is *if all techniques have random matrices and if each associated labour vector is such that* $\bar{v}$ *is small enough, we get an approximate surrogate production function, whatever numéraire is used. The deviation of relative prices from relative values can be large, without affecting this result, provided only the mean of* **v** *is small.*

We mention in passing that systems with a relatively simple wage curve will have prices that are relatively simple as functions of the rate of profit. Since we are mainly interested in the production function, we here only show that standard prices are linear functions of the rate of profit, if the non-dominant eigenvalues are small. Let the labour vector again be represented as a linear combination $\mathbf{l}=\gamma_1\mathbf{x}_1+\ldots+\gamma_n\mathbf{x}_n$ of the right-hand side eigenvectors $\mathbf{x}_i$, $(1+R_i)\mathbf{A}\mathbf{x}_i=\mathbf{x}_i$ (the strong normalisation is here not needed). Then we obtain for standard prices by a transformation analogous to that of Section 3, with $R = R_1$:

$$\bar{\mathbf{p}}=\left(1-\frac{r}{R}\right)(\mathbf{I}-(1+r)\mathbf{A})^{-1}\sum\gamma_i\mathbf{x}_i=\frac{R_1-r}{R_1}\sum_{i=1}^{n}\frac{1+R_i}{R_i-r}\gamma_i\mathbf{x}_i,$$

hence, if $R_2,\ldots,R_n$ tend to infinity:

$$\bar{\mathbf{p}}(r)=\frac{1+R}{R}\gamma_1\mathbf{x}_1+\sum_{i=2}^{n}\left(1-\frac{r}{R}\right)\gamma_i\mathbf{x}_i.$$

This formula allows us to interpret $\bar{\mathbf{p}}(r)$ as a linear function of the extreme values $\bar{\mathbf{p}}(\mathrm{R}) = [(1+\mathrm{R})/\mathrm{R}]\gamma_1 \mathrm{x}_1$ and labour values **u**, $\mathbf{u} = \bar{\mathbf{p}}(0) = \bar{\mathbf{p}}(R) + \gamma_2 \mathbf{x}_2 + \ldots + \gamma_n \mathbf{x}_n$:

$$\bar{\mathbf{p}}(r) = \bar{\mathbf{p}}(R) + \left(1 - \frac{r}{R}\right)(\bar{\mathbf{p}}(0) - \bar{\mathbf{p}}(R)),$$

therefore

$$\bar{\mathbf{p}}(r) = \mathbf{u} + \frac{r}{R}(\bar{\mathbf{p}}(R) - \mathbf{u}).$$

Such linear deviations of prices from values were empirically observed and discussed by Anwar Shaikh (1998) and Mariolis and Tsoulfidis (2009).[12] The observation that prices are near-linear functions of the rate of profit, as found by these authors, and the near-linear wage curves that have appeared frequently in the empirical literature since Krelle (1976), can thus be explained in terms of our assumptions.

But why should we expect non-dominant eigenvalues to be small in a large class of systems? Randomness is only a sufficient condition. A complete mathematical answer as to the necessity would presuppose a satisfactory solution to the inverse eigenvalue problem, applied to the whole spectrum of eigenvalues of a semi-positive matrix (Minc,

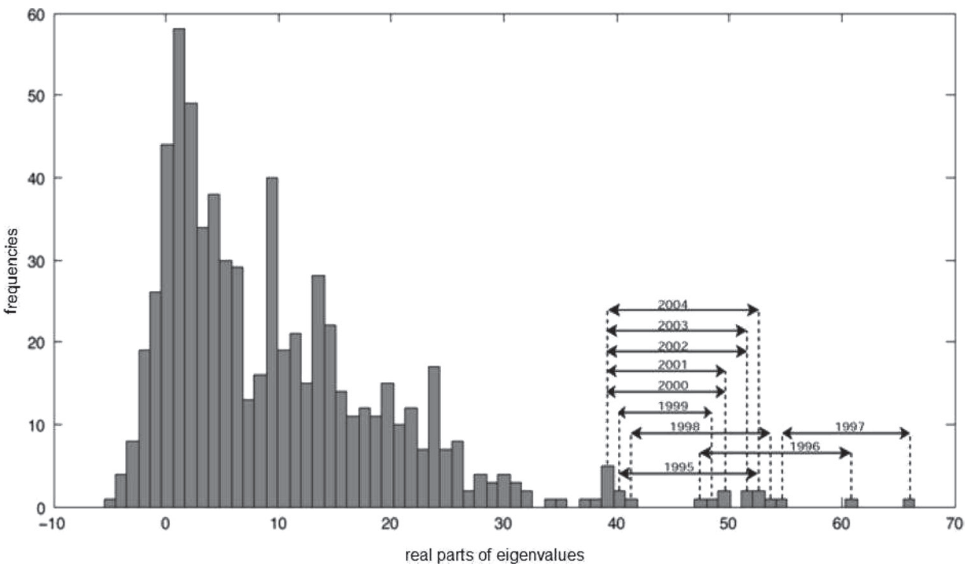

Fig. 2. *The empirical distribution of the eigenvalues of 10 input–output tables for Germany, 1995–2004. The arrows indicate the distance between the dominant eigenvalue and the largest (in terms of its real part) non-dominant eigenvalue of the corresponding table. Source:* Statistisches Bundesamt (2007): Fachserie 18, Volkswirtschaftliche Gesamtrechnungen, Reihe 2, Input–Output-Rechnung 1995–2004, Wiesbaden. *Notes:* All tables are in current prices for input coefficients. Classification of goods according to the European Standard of 2002. Three sectors were omitted, for which all coefficients were zero in the case of Germany. The 71 × 71 tables therefore were reduced to 68 × 68. All matrices have rank 68.

12 They note the condition $rk\mathbf{A} = 1$, to be discussed in Section 5.

1988, p. 183). Some heuristic considerations were offered in Schefold (2008B), where it was shown by means of a counterexample that the non-dominant eigenvalues do not generally tend more rapidly to zero for larger systems than for smaller ones. Hence, we seem compelled provisionally to infer that the decisive property leading to non-dominant eigenvalue is randomness, not so much the dimension of the matrix.

Nevertheless, there are deterministic counterparts to the random matrices to which we turn in the next section. We, here, present a diagram showing the empirical distribution of the real parts of the eigenvalues of 10 input–output tables for Germany (Figure 2).

The tendency for non-dominant eigenvalues to cluster around zero is visible. It becomes more visible if the moduli are represented, but numerical experiments with randomly generated matrices lead to a still more pronounced clustering. For a recent empirical analysis with similar results, see Mariolis and Tsoulfidis (2010). Their results confirm what this diagram suggests: only a few eigenvalues are not much smaller than *dom***A**—here, less than 2% are in the range between *dom***A**/2 and *dom***A** = 1/(1 + *R*). This observation leads to a plausible explanation of the small 'wiggles'. If $R_2,\ldots,R_h$ are finite, $|R_i| > R$ (neglecting imprimitive matrices with some $|R_i| = R$, $i \neq 1$), while $R_h,\ldots,R_n$ are large enough to be ignored, the wage curve $\tilde{w}$ becomes, using the strong normalisation:

$$\frac{1}{\tilde{w}} = \frac{1+R}{R-r}\mathbf{q}_1\mathbf{x}_1 + \sum_{i=2}^{h}\frac{1+R_i}{R_i-r}\mathbf{q}_i\mathbf{x}_i + \sum_{i=h+1}^{n}\mathbf{q}_i\mathbf{x}_i .$$

If h is small enough and n large, the deviations $\mathbf{d}-(\mathbf{q}_1+\ldots+\mathbf{q}_h)=\mathbf{q}_{h+1}+\ldots+\mathbf{q}_n$ and $\mathbf{1}-(\mathbf{x}_1+\ldots+\mathbf{x}_h)=\mathbf{x}_{h+1}+\ldots+\mathbf{x}_n$ may again be considered as independent and the absolute value of the mean of $\mathbf{x}_{h+1}+\ldots+\mathbf{x}_n$ as small, so that we may neglect the last term in the formula for $\tilde{w}$. The deviations of the wage curve $\tilde{w}$ from linearity (the 'wiggles'), then, are due to the middle term. We get for $0 \leq r < R$:

$$\left|\frac{1}{\tilde{w}} - \frac{1+R}{R-r}\mathbf{q}_1\mathbf{x}_1\right| \leq \sum_{i=2}^{h}\left|\frac{1+R_i}{R_i-r}\right|\left|\mathbf{q}_i\mathbf{x}_i\right|.$$

'Wiggles' thus can arise if some $\mathbf{q}_i\mathbf{x}_i$; $i = 2,\ldots,h$; are not zero, but their amplitudes will not be large if prices are close to values and/or the numéraire is close to the standard. Assuming (for simplicity) that $R_i > 0$; $i = 2,\ldots,h$; we see that linearity will be disturbed the more, the smaller $R_i - R > 0$ and the closer r to R. The linearity of the wage curve, thus, is more questionable at higher rates of profit (but the deviation from linearity remains bounded). This explains why reverse capital deepening and the paradox of paradoxes could be observed rarely in Han and Schefold (2006) and why the macro effects of the paradoxes remained small.

What can we learn if $h = n$, i.e. if there is no concentration of the eigenvalues near zero? Of course, the wage curves will still be linear in the standard commodity case and if the labour theory of value holds, but there is then only one case in which it will be sufficient that $\bar{m} = 0$ or $\bar{v} = 0$: if all non-dominant eigenvalues are equal to some $R_2 > R$, then we can write:

$$\frac{1}{\tilde{w}} - \frac{1+R}{R-r}\mathbf{q}_1\mathbf{x}_1 = \frac{1+R_2}{R_2-r}(\mathbf{q}_2\mathbf{x}_2+\ldots+\mathbf{q}_n\mathbf{x}_n)$$

and $\mathbf{q}_2\mathbf{x}_2 + \ldots + \mathbf{q}_n\mathbf{x}_n = \mathbf{mn} = n\bar{m}\bar{v}$. But it is *not* sufficient that the labour theory of value holds 'on average', i.e. that $\bar{v} = 0$, if the $R_2, \ldots, R_n$ are neither large nor equal. For even if we assume $R < R_2 < |R_i|$; $i = 3, \ldots, n$;—which seems to be the best we can do—we get:

$$\left| \frac{1}{\tilde{w}} - \frac{1+R}{R-r} \mathbf{q}_1\mathbf{x}_1 \right| \leq \frac{1+R_2}{R_2 - r} \left(|\mathbf{q}_2\mathbf{x}_2| + \ldots + |\mathbf{q}_n\mathbf{x}_n| \right).$$

Since absolute values have come in, positive and negative deviations do not cancel. If the wage curve is to be linear in this case, prices must be close to labour values (the $|\mathbf{x}_i|$ are small; $i = 2, \ldots, n$) or the numéraire close to the standard (the $|\mathbf{q}_i|$ are small; $i = 2, \ldots, n$), but it is not sufficient any more that these properties hold only on average, since the non-dominant eigenvalues are not small.

5. One-industry systems with strictly linear wage curves

The general form of semi-positive indecomposable input–output matrices **A** with vanishing non-dominant eigenvalues is

$$\mathbf{A} = \mathbf{cf},$$

where **c** is a positive column vector and **f** a positive row vector, as can easily be shown.[13] These matrices, with a labour vector, are here called *one-industry systems*, because **f**, the composition of capital (as we name it), is the same in all sectors; the components of **c** indicate how this capital is distributed.[14] If **f** = **e**, as is the case (but only on average!) for random matrices, we can speak of an even composition of capital. The book of blueprints consists of a finite number of such matrices and associated labour vectors. This can be extended and random processes can here be introduced by assuming that we start from given systems with **A** = **cf**, hence with non-dominant eigenvalues equal to zero, and by regarding the spectrum of available techniques as a finite number of perturbations of these systems such that the non-dominant eigenvalues remain small. This would represent a true generalisation, compared with the random matrices considered in Section 3, since the rows would not even on average be proportional to **e**; however, we must use this generalisation with caution, since a rigorous mathematical theory determining the admissible extent of the perturbations, e.g. in terms of the admissible variance of the elements of a perturbed input matrix, does not seem to be available (the non-dominant eigenvalues obviously can cease to be small if no conditions are imposed on the perturbations). The procedure means that we experiment with a compromise between randomisation for the representation of large systems (the statistical view) and the determination of the individual structure of production.

13 The only non-zero eigenvalue of **cf** is **fc**, the Frobenius eigenvalue $(1 + R)^{-1}$ of **cf**, since $\mathbf{qcf} = \lambda\mathbf{q}$ implies either **f** = **q** and $\lambda = \mathbf{qc}$ or $\lambda = 0$ and $\mathbf{qc} = 0$. Conversely, if **A** has $n - 1$ vanishing eigenvalues, **A** maps $\mathfrak{R}^n$ on a linear subspace, $rk\mathbf{A} = 1$ and all rows of **A** are proportional.

14 One-industry systems can be transformed into systems in which industries $2, \ldots, n$ seem to use no inputs other than labour, but they are neither non-basic nor equivalent to the centre of fixed capital systems (Schefold, 1989 [1971], p. 147); they represent a novelty in the Sraffa literature. The transformation is (for $R = 0$) obtained by putting $\bar{\mathbf{A}} = \mathbf{QA}$, $\bar{\mathbf{l}} = \mathbf{Ql}$, **Q** output matrix with $q_{1i} = -c_i / c_1$; $i = 2, \ldots, n$; and $q_{ij} = \delta_{ij}$ otherwise.

Physics is sometimes said to be most difficult where quantum mechanics and classical mechanics meet, but our approach in the following model is very simple: we assume that the perturbations are small enough.[15]

If **A** is productive with $R > 0$, we have $\mathbf{fA} = \mathbf{fcf}$; hence, **f** is the Frobenius eigenvector, $1 + R = 1/\mathbf{fc}$ and $\mathbf{fc} = dom\mathbf{A} < 1$. The price equations are

$$\mathbf{p} = (1 + r)\mathbf{cfp} + w\mathbf{l}$$

and the price vector $\bar{\mathbf{p}}$ in terms of **f** is irregular as a function of r. We get

$$1 = \mathbf{f}\bar{\mathbf{p}} = (1+r)\mathbf{fcf}\bar{\mathbf{p}} + \bar{w}\mathbf{fl},$$

therefore linear functions for wage curve and standard prices:

$$\bar{w} = \frac{1}{\mathbf{fl}}\left(1-(1+r)\mathbf{fc}\right) = \frac{1}{\mathbf{fl}}\left(1-\frac{1+r}{1+R}\right) = \frac{R-r}{(1+R)\mathbf{fl}},$$

$$\bar{\mathbf{p}} = (1+r)\mathbf{c} + \frac{R-r}{(1+R)\mathbf{fl}}\mathbf{l},$$

hence[16]

$$\hat{\mathbf{p}} = \frac{\bar{\mathbf{p}}}{\bar{w}} = \mathbf{l} + \mathbf{fl}(1+R)\frac{1+r}{R-r}\mathbf{c},$$

and the wage curve for any numéraire $\mathbf{d} \geq 0$ results:

$$w = \frac{1}{\mathbf{d}\hat{\mathbf{p}}} = \frac{R-r}{(R-r)\mathbf{dl} + \mathbf{fl}(1+R)(1+r)\mathbf{dc}} = \frac{R-r}{R\mathbf{dl} + \mathbf{fl}(1+R)\mathbf{dc} + r\left(\mathbf{fl}(1+R)\mathbf{dc} - \mathbf{dl}\right)}.$$

This is in essence the same familiar hyperbola that we obtained for random matrices in Section 3 and discussed in Section 4. The hyperbola becomes linear for (i) $\mathbf{d} = \mathbf{f}$ (standard numéraire), (ii) for $(1 + R)\mathbf{c} = \mathbf{c}/\mathbf{fc} = \mathbf{l}/\mathbf{fl}$ (labour theory) or, more generally, (iii) for $\mathbf{d}(\mathbf{fl}(1 + R)\mathbf{c} - \mathbf{l}) = 0$—this condition means that the scope of the systems generating linear wage curves is considerably enlarged: **d** can be any point on the intersection of an $(n - 1)$-dimensional hyperplane and the positive orthant—or

$$(1 + R)\mathbf{dc} = \mathbf{dc}/\mathbf{fc} = \mathbf{dl}/\mathbf{fl}.$$

[15] We can also say that a sufficient condition for the wage curves to be nearly hyperbolic and for the non-dominant eigenvalues to be small is that the matrix is random; the rows then are not proportional (except on average), but the coefficients must be i.i.d. Another sufficient condition is that the rows are proportional, but that the distribution of the coefficients on the row is determinate and can be arbitrary. We can then add the hypothesis that perturbations of $\mathbf{A} = \mathbf{cf}$ will, within certain limits, keep the non-dominant eigenvalues small, but we do not know what these limits are, how they are related to the average distribution of the coefficients in the rows and how they have to be formulated so as to obtain a family of techniques. In contrast, it is easy to formulate a sufficient condition for one-industry systems without perturbations to constitute a family: **f** must be kept constant (cf. the one-industry systems of constant capital composition considered below).

[16] Note the irregular character of the price movement. The vector $\bar{\mathbf{p}}$ remains in a two-dimensional hyperplane, while prices of regular systems move in n dimensions: a difference which comes to the fore as soon as $n \geq 3$ (two-dimensional examples often do not reflect the full complexity of capital theory, as the paradox of paradoxes, referred to above in note 5, also shows).

It must again turn out that the properties of prices being near to labour values, of the numéraire being close to the standard and of eigenvalues being close to zero reinforce each other. Because we have a determinate system, only the first two properties matter; by a procedure analogous to that employed in Section 3—but the formulas are more complex—one confirms this proposition. The property is of lesser interest here since, if the composition of capital is constant, it is natural to choose it as the numéraire, as we shall do below.

If we were only interested in finding, for a given system with a standard numéraire and hence a linear wage curve, any other system that would also have the same standard and hence also a linear wage curve, we should have a large variety of matrices to choose from,[17] but the combinations of the methods of the two systems would not in general have the same standard numéraire. What we need is a family of techniques such that combinations of the methods of different techniques yield techniques of the same family: this compels us to focus on a given composition of capital.

We are interested in the conditions under which technical changes within the same family of systems will leave the wage curves straight. The change can affect the labour vector and/or the input–output matrix, and if the latter, it can in principle affect **c** or **f**. We continue to interpret **f** as the *composition of capital* which remains the same for all activity levels **q**, given **A**, since $\mathbf{qA} = \mathbf{q}(\mathbf{cf}) = (\mathbf{qc})\mathbf{f}$ varies only with the total volume **qc**. We interpret **c** as the *distribution of capital* over industries, since the total volume **qc** of the capital goods of composition **f** is distributed in proportion c_i to the inputs $\mathbf{a}_i$ of industry *i*. Note that c_i can also be interpreted as an index of productivity (the smaller c_i, the smaller the proportion of **f** required to produce one unit of commodity *i*), and c_i can, with **A** considered as random, still be interpreted as a mean pertaining to industry inputs $\mathbf{a}_i$, if $\mathbf{a}_i = c_i\mathbf{e}$, but c_i is not to be confused with an activity level: it characterises the inputs relative to the output and is not, as an activity level would be, a common multiplier for both.

With a given capital *composition*, all industries are thought to be somewhat alike (equal apart from random perturbations)—around the year 1900, steel is important in each sector; electronics is important around 2000. On the other hand, some idea of a physical capital–labour ratio is associated with every method, hence—with labour not random—the idea of a given *distribution* of capital. We now assume that technical change affects the methods employed in different industries, say in industry *i*, by a change of the distribution of capital c_i or the labour input l_i, but that the composition of capital does not change.[18] This assumption defines a family of systems.

We start from a given system in this family, assuming that the wage curve happens to be linear. If this is the case because of the most general condition (iii) above, i.e. because $\mathbf{dc}/\mathbf{fc} = \mathbf{dl}/\mathbf{fl}$, without assuming that the numéraire is proportional to the Frobenius vector **f**, technical change will leave the wage curve straight only if a proportionate change of **c** and/or **l** takes place. For example, all components of **c** rise and **c** is replaced by $\bar{\mathbf{c}} = \alpha\mathbf{c}$, $\alpha > 1$ and **l** is reduced, with $\bar{\mathbf{l}} = \beta\mathbf{l}$, $0 < \beta < 1$. This could be a process of technical change as mechanisation, taking place in time at a given rate of profit: a process of accumulation with technical progress as in classical theory. Or we could have different techniques for different levels of α and β, available at the same

[17] The vector space of matrices **A** of order n has dimensions n^2; it contains a subspace of dimension $n^2 - n + 1$ of matrices **A** having the same Frobenius eigenvector **d** in common.

[18] Slight perturbations of the coefficients will not cause the non-dominant eigenvalues to differ much from zero, because of the continuous dependence of the roots of the characteristic polynomial on its coefficients.

time, as in neoclassical theory, and in this case the different straight wage curves would seem to correspond to a surrogate production function reflecting the possibility of substitution, hence of choosing between different degrees of mechanisation that would be optimal at different levels of distribution (the wage curves of less-mechanised techniques would appear on the envelope at higher rates of profit).

However, the construction would not be generally valid for the reason encountered in Section 2: even if only the techniques represented by $\mathbf{c}$ and $\bar{\mathbf{c}}$, $\mathbf{l}$ and $\bar{\mathbf{l}}$ and hence apparently only two wage curves were given, it would be possible to combine the methods of both. These combinations would give rise to many more wage curves and these would, in general, not be straight. A similar argument could be made if the wage curve of the system from which we start would be straight because of condition (ii).

But matters are different for matrices of the family of one-industry matrices of stable capital composition, if the numéraire is chosen according to condition (i). We then have $\mathbf{d} = \mathbf{f}$ and the term $\mathbf{fl}(1 + R)\mathbf{dc} - \mathbf{dl}$, which causes the hyperbolic form of the wage curve, becomes $(\mathbf{fl}/\mathbf{fc})\mathbf{fc} - \mathbf{fl}$. The latter expression vanishes for all $\mathbf{c}$ and $\mathbf{l}$. The wage curve therefore is linear and may be written as

$$\bar{w} = \frac{1}{\mathbf{fl}}\left(1-(1+r)\,\mathbf{fc}\right),$$

with $\bar{w}(0)=(1-\mathbf{fc})\,\mathbf{fl}$, $\bar{w}\,(\mathrm{R})=0$, $\mathrm{R}=(1/\,\mathbf{fc})-1$. The wage curve remains straight for any changes of $\mathbf{c}$ and $\mathbf{l}$ and for all combinations of processes, and the position of each wage curve can be defined by calculating the corresponding $\bar{w}(0)$ and R. This family thus gives rise to a rigorous *surrogate production function*, because the analysis is limited to one-industry systems and $\mathbf{f}$ remains *strictly* constant. The construction proves that surrogate production functions exist not only in the case of one-commodity systems.

We can generalise, dealing with an *approximate* surrogate production function, if changes of $\mathbf{f}$ are small because of corresponding random perturbations of the methods of production. Our argument of the stable capital composition then is justified by the slow movements of averages of large systems. The construction is not as unrealistic as it may seem, in spite of the obvious heterogeneity of the compositions of capital in any small number of industries picked out at random from the empirical input–output table of an actual economy, to the extent that the perturbations of $\mathbf{A}$ leave the non-dominant eigenvalues small. Because of the trend assumed for $\mathbf{f}$, the capital composition, we might speak of *random one-industry systems*. In the case of random matrices, the variance of the coefficients of the input–output matrix can be as large as c/n^2 (c constant), according to the theorem by Goldberg and Neumann (2003), which means that the individual coefficients a_{ij} of row $\mathbf{a}_j$ can deviate from the mean of $\mathbf{a}_j$ by considerable amounts. Similarly, considerable perturbations of the coefficients in random one-industry systems might be compatible with the preservation of the essential properties of such systems.

6. Conclusions

This paper has been written with the intention of taking up the challenge posed by the contrast between the claim of the Cambridge critique to have successfully undermined neoclassical theory by means of the discovery of the paradoxes of capital theory and the empirical finding that these paradoxes appear to be rare—an appearance that seems to confirm Joan Robinson's treatment of them as mere 'curiosa'

(Robinson, 1969 [1956], p. 109). In the course of the exposition of the problem (Section 1), we found that almost linear wage curves of individual techniques were shown to be both sufficient and also essentially necessary for the construction of approximate surrogate production functions. Marginal products can then be said to exist, the paradoxes being absent or of small importance. Five results concerning the character of the wage curves may be enumerated:

(1) The paradoxes are easy to generate, if only non-basics or 'Austrian' processes are involved. The analysis of the closest analogon of 'Austrian' processes among basic systems, the family of circular systems, revealed that the curvature of wage curves and hence the magnitude of declination may still become arbitrarily large, but the direct confrontation of the wage curves so constructed is not licit because of combinations of processes. The paradoxes do not disappear in consequence, but their likelihood is diminished (Section 2 and Schefold (2008B)).

(2) Large random systems, the second family considered, generate approximately hyperbolic curves, because the non-dominant eigenvalues tend to zero. This property, reinforced by numéraires close to the standard, leads to near-linear wage curves (Sections 3 and 4). There is empirical evidence for the clustering of the non-dominant eigenvalues of input–output tables near zero. This indicates a random character of actual techniques. The results could be strengthened considerably by looking at the vector of deviations **m** of the numéraire vector **d** from the 'strongly normalised' (Section 3) left-hand eigenvector of input matrix **A** and at the vector of deviations **v** of the labour vector from the strongly normalised right-hand eigenvector of **A**. If the covariance of the components of **m** and **v** is zero and if **A** is random, the wage curve will be linear if $\bar{m}$, the mean of the components of **m**, or $\bar{v}$, the mean of the components of **v**, is zero. Hence, an approximate surrogate production function exists for techniques for which the matrices are random and the associated $\bar{v}$ small enough, whatever numéraire is chosen. The wage curves will then be nearly linear, although prices may differ from values with a large variance and although the numéraire is not standard (Section 4).

(3) The deterministic counterpart of random systems consists of the family of one-industry systems. They lead to hyperbolic wage curves. One can show, as in the case of random systems, that the properties of prices being close to values and of the numéraire being close to the standard reinforce each other so as to generate almost linear wage curves (Section 5). Hence, irregular Sraffa systems, though not generic, are relevant as approximations or attractors, to which actual systems gravitate to the extent that they are random or—choosing now a loose expression—tend to random deviations (with zero mean) from equal compositions of capital.

(4) The one-industry systems of stable capital composition form a family for which rigorous surrogate production functions exist. They may also be used to represent the classical process of accumulation with mechanisation. If technical change takes the form of perturbations of one-industry systems with a slow change of the capital composition (near random systems), the wage curves remain approximately straight and an approximate surrogate production function exists (Section 5).

(5) The results provide a theoretical explanation for the empirical finding in Han and Schefold (2006) that reverse capital deepening and reverse substitutions of labour can exist but must be rare. For if real systems are approximately random but not strictly random, and if the numéraire is near but not equal to the standard

commodity, wage curves are nearly linear. However, the existence of a few non-dominant eigenvalues that are not small can lead to 'wiggles' (Section 4) of the wage curves, such that two individual wage curves may occasionally intersect not only more than once but more than twice (the paradox of paradoxes of Section 1).

These findings have implications for the interpretation of classical theory and Marx; I leave that for another paper. One hint only: the effect of the numéraire drift (Section 4) is a big problem, if one wants to construct a production function not only in the neighbourhood of a given technique (with the numéraire proportional to the standard commodity of that technique), but, as is usually the aim, for the entire possible range of the rate of profit. In contrast, the Marxian assertions about aggregates primarily concern the valuation of individual techniques, but comparisons come in with the influence of the organic composition of capital on accumulation. We are thus led to the general conclusion that the 'local' construction of the production function is not devoid of content: capital can be aggregated in more general cases than one-commodity systems. Although the means of production are heterogeneous and themselves produced in an interdependent system, the aggregate capital, combined with labour, yields an aggregate output such that diminishing partial returns result. The construction is possible by a kind of statistical smoothing, although the individual techniques are linear, of the fixed coefficients variety, with constant returns to scale. The problem formulated by Hicks (1932)—how is marginal productivity to be reconciled with fixed coefficients of production?—and for which Samuelson found his ingenious but incomplete solution, can, within limits, be approached successfully in a stochastic setting.

The construction seems sufficiently robust to support contentions such as those that I associate with Böhm-Bawerk (1914): suppose the book of blueprints is given and does not change, suppose both factors are fully employed, suppose that trade unions enforce a rise of real wages. The choice of techniques, guided by mere profit maximisation, will then lead to an increase in the intensity of capital and hence to unemployment. Can it be cured by Keynesian means, either because of a demand effect resulting from an increase in wages or by state expenditure, while real wages stay at their elevated level? The answer is 'no', since full employment of 'capital' was assumed; full employment requires a lowering of real wages in this case—the Keynesian remedy, which presupposes idle capacity, is not applicable. The idea of rigidly given levels of capacity, the stationary nature of the economy and other, hidden assumptions of this story may be questioned, but that leads into different territory. Whatever other arguments may be adduced, the point is that this argument can no longer be simply dismissed *on the basis of the critique of capital*.

We have confirmed, on the other hand, that the production function is not grounded on foundations that are both rigorous and general, and that our less rigorous construction, with its introduction of a statistical notion, randomness, does not support the whole edifice built on the production function. In particular, *constant and stable elasticities of substitution found no support in this investigation*. The theory of normal prices, with the wage curve as its main tool, emerges as the fundamental concept, and the aggregate production function is a derived concept of limited applicability. For example, if the above problem of Böhm-Bawerk's is posed, its solution may be sought directly by visualising the rise in real wages in the diagram of wage curves and by determining the more capital-intensive technique there, with the advantage of rendering the problem of the transition to the new technique more explicit.

An accompanying paper (Schefold, 2013) shows that the number of wage curves that appear on the envelope is surprisingly small. If we think of the techniques as given by input–output tables for 100 industries and if there are 10 feasible methods of production in each industry, there will be $s = 10^{100}$ wage curves, but under assumptions similar to those made here, fewer than $\ln s = 100 \ln 10 \approx 230$ are likely to appear on the envelope. There will not be as many switchpoints on the envelope as one might have expected, in accordance with the results found in Han and Schefold (2006), and the pseudoproduction function cannot be as smooth as postulated. This confirms that the system of wage curves is the essential analytical tool.

However, there remains the more fundamental point that has been established in this paper: the inverse relationship between the rate of profit and the intensity of capital which holds in most cases; theoretical reasons have been given as to why the paradoxes of capital are rare, in accordance with our earlier empirical findings. The consequences for macroeconomics, based on a realistic book of blueprints, remain to be drawn.

Bibliography

Bidard, C. and Schatteman, T. 2001. The spectrum of random matrices, *Economic Systems Research*, vol. 13, no. 3, 289–98

Böhm-Bawerk, E. von. 1914. Macht oder ökonomisches Gesetz? *Zeitschrift für Volkswirtschaft, Socialpolitik und Verwaltung*, vol. 23, 205–71

Cohen, A. J. and Harcourt, G. C. 2003. Whatever happened to the Cambridge capital theory controversies? *Journal of Economic Perspectives*, vol. 17, no. 1, 199–214; (Comments in vol. 17, no. 4)

Gallaway, L. and Shukla, V. 1976. The neoclassical production function, *American Economic Review*, vol. 64, no. 3, 348–58

Gantmacher, F.R. 1966 [1959]. *Matrizenrechnung, Teil II*, Hochschulbücher für Mathematik 37, Berlin, VEB Deutscher Verlag der Wissenschaften

Garegnani, P. 1970. Heterogeneous capital, the production function and the theory of distribution, *Review of Economic Studies*, vol. 37, 407–36

Garegnani, P. 2003. Savings, investment and capital in a system of general intertemporal equilibrium (with 2 appendices and a mathematical note by M. Tucci), pp. 117–75 in Petri, F. and Hahn, F. (eds), *General Equilibrium. Problems and Prospects*, London, Routledge

Goldberg, G. and Neumann, M. 2003. Distribution of subdominant eigenvalues of matrices with random rows, *SIAM Journal of Matrix Analysis and Applications*, vol. 24, no. 3, 747–61

Goldberg, G., Okunov, P., Neumann, M. and Schneider, H. 2000. Distribution of subdominant eigenvalues of random matrices, *Methodology and Computing in Applied Probability*, vol. 2, no. 2, 137–51

Gröbner, W. 1966. *Matrizenrechnung*, Mannheim, Bibliographisches Institut

Han, Z. and Schefold, B. 2006. An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory, *Cambridge Journal of Economics*, vol. 30, no. 5, 737–65

Harcourt, G. C. 1972. *Some Cambridge Controversies in the Theory of Capital*, Cambridge, UK, Cambridge University Press

Hicks, J. R. 1932. Marginal productivity and the principle of variation, *Economica*, vol. 35, 79–88

Krelle, W. 1976. Basic facts in a capital theory. Some lessons from the controversy in capital theory, *Revue d'économie politique*, vol. 87, 283–329

Levhari, D. 1965. A nonsubstitution theorem and switching of techniques, *Quarterly Journal of Economics*, vol. 79, 98–105

Levhari, D. and Samuelson, P. A. 1966. The reswitching theorem is false, *Quarterly Journal of Economics*, vol. 80, 518–19

Mariolis, T. and Tsoulfidis, L. 2009. Decomposing the changes in production prices into 'capital-intensity' and 'price' effects: theory and evidence from the Chinese economy, *Contributions to Political Economy*, vol. 28, no. 1, 1–22

Mariolis, T. and Tsoulfidis, L. 2010. 'Eigenvalue Distribution and the Production Price-Profit Relationship in Linear Single Product Systems: Theory and Empirical Evidence', University of Macedonia Discussion Paper no. 16

Minc, H. 1988. *Nonnegative Matrices*, Wiley–Interscience series in Discrete Mathematics and Optimization, New York, John Wiley

Pasinetti, L. L. 1966. Changes in the rate of profit and switches of techniques, *Quarterly Journal of Economics*, vol. 80, 503–17

Robinson, J. 1953–54. Production function and the theory of capital, *Review of Economic Studies*, vol. 21, no. 2, 81–106

Robinson, J. 1969 [1956]. *The Accumulation of Capital*, London, Macmillan

Salvadori, N. and Steedman, I. 1988. No reswitching? No switching! *Cambridge Journal of Economics*, vol. 12, 481–546

Samuelson, P. A. 1962. Parable and realism in capital theory: the surrogate production function, *Review of Economic Studies*, vol. 29, 193–206

Schefold, B. 1976. Relative prices as a function of the rate of profit, *Zeitschrift für Nationalökonomie*, vol. 36, 21–48

Schefold, B. 1989 [1971]. *Mr. Sraffa on Joint Production and Other Essays*, London, Unwin & Hyman

Schefold, B. 1997. *Normal Prices, Technical Change and Accumulation*, Studies in Political Economy, London, Macmillan

Schefold, B. 1999. The nature of capital and interest, pp. 241–62 in Loef, H.-E. and Monissen, H. G. (eds), *The Economics of Irving Fisher. Reviewing the Scientific Work of a Great Economist*, Cheltenham, Elgar (Comment by V. Caspari, pp. 263–7)

Schefold, B. 2004. *Beiträge zur ökonomischen Dogmengeschichte*, ed. by V. Caspari, Düsseldorf, Verlag Wirtschaft und Finanzen

Schefold, B. 2005. Reswitching as a cause of instability of intertemporal equilibrium, *Metroeconomica*, vol. 56, no. 4, 438–76

Schefold, B. 2006. C.E.S. production functions in the light of the Cambridge critique, *Journal of Macroeconomics*, vol. 30, 783–97

Schefold, B. 2008A. Savings, investment and capital in a system of general intertemporal equilibrium—an extended comment on Garegnani with a note on Parrinello (Mai 2006), pp. 127–84 in Chiodi, G. and Ditta, L. (eds), *Sraffa or an Alternative Economics*, Basingstoke, Macmillan

Schefold, B. 2008B. Families of strongly curved and of nearly linear wage curves: a contribution to the debate about the surrogate production function, *Bulletin of Political Economy*, vol. 2, no. 1, 1–24

Schefold B. 2012. A comment on 'The measurements of prices of production' by Peter Flaschel, Reiner Franke & Roberto Veneziani, *Review of Political Economy*, vol. 24, no. 3, 437–44

Schefold, B. 2013. Only a few techniques matter! On the number of curves on the wage frontier, to be published (Proceedings of the Sraffa Conference, Rome 2010) Levrero, E. S., Palumbo, A. and Stirati, A. (eds), *Sraffa and the Reconstruction of Economic Theory*, vol. I, *Theories of Value and Distribution*, Basingstoke, Palgrave Macmillan

Sen, A. 1974. On some debates in capital theory, *Economica*, vol. 41, no. 163, 328–35

Shaikh, A. 1987. Humbug production function, pp. 690–1 in Eatwell, J., Milgate, M. and Newman, P. (eds) *The New Palgrave. A Dictionary of Economics*, vol. 2, London, Macmillan

Shaikh, A. 1998. The empirical strength of the labour theory of value, pp. 225–51 in Bellofiore, R. (ed.), *Marxian Economics. A Reappraisal*, vol. 2, Basingstoke, Macmillan

Solow, R. 1987. Second thoughts on growth theory, pp. 13–28 in Steinherr, A. and Weiserbs, D. (eds), *Employment and Growth: Issues for the 1980s*, Dordrecht, Kluwer

Sraffa, P. 1960. *Production of Commodities by Means of Commodities. Prelude to a Critique of Economic Theory*, Cambridge, UK, Cambridge University Press

Steedman, I. and Tomkins, J. M. 1998. On measuring the deviation of prices from values, *Cambridge Journal of Economics*, vol. 22, 379–85

Appendix: Random matrices

It has been asserted in the paper that the non-dominant eigenvalues of large random matrices are small. It will help the understanding of this proposition if intuitive reasons for the result are given, before we formulate the actual mathematical theorem.

Let a_{ij}; $i, j = 1, \ldots, n$; denote the elements of the semi-positive indecomposable matrix **A**; $n = 1, 2, \ldots$. The a_{ij} are random variables, i.i.d., with mean μ. The averages of each row and each column of **A** will tend to μ for large n because of the strong law of large numbers. We therefore get $\left|\mathbf{e}\mathbf{a}^j / n - \mu\right| < \varepsilon > 0$ (**e** is the summation vector and $\mathbf{a}^j$ is the column of **A**), so that the column average of the inputs in each industry differs from μ only by a given ε, if n is large enough. This implies that the column sums approach $n\mu$ and tend, thus, to be equal if n increases so slowly that $n\varepsilon$ can be made to go to zero. Hence, *dom***A** tends to $n\mu$ and **e** is the Frobenius eigenvector of **A**. Given the distribution of the elements, *dom***A** increases with n for given μ, but it is more instructive to assume $\mu = \lambda/n$, with *dom***A** tending to λ.

To get an idea why all other eigenvalues will tend zero, we define $\mathbf{q}_i = \mathbf{e}_1 - \mathbf{e}_i$, $\mathbf{e}_i$; $i = 1, \ldots, n$; being the unit (row) vectors, and we obtain $\mathbf{q}_i\mathbf{A} = \mathbf{a}_1 - \mathbf{a}_i$; $i = 2, \ldots, n$. The elements of $\mathbf{q}_i\mathbf{A}$ will nearly be normally distributed with mean zero because of the central limit theorem; $\left|\mathbf{q}_i\mathbf{A}\right|$ will be small if the variance of the elements of $\mathbf{a}_1 - \mathbf{a}_i$ is small. **A** will then, for large n, be close to matrix $\mu\mathbf{E}$ (**E** is the matrix with all elements equal to one); the $\mathbf{q}_i$ are eigenvectors of $\mu\mathbf{E}$ with eigenvalues equal to zero; $i = 2, \ldots, n$. The proof of Bidard and Schatteman (2001) does not require the small variance argument and ensures convergence by having recourse to higher moments of the distribution.

A rigorous mathematical statement was independently given by Goldberg *et al.* (2000) and this has been generalised significantly by Goldberg and Neumann (2003). The latter theorem is essentially as follows (Goldberg and Neumann, 2003, p. 749). The elements of **A** are random with mean $1/n$ and the rows of **A** independent. The variance of $b_{ij} = a_{ij} - (1/n)$ is bounded by c/n^2 and the absolute value of the covariance of any two rows b_i, b_j, $i \neq j$, is bounded by c/n^3, c constant. For $0 < \delta < 1$, $0 < p < 1$ there is $N(\delta, p)$, such that for any $n > N(\delta, p)$ and for any γ with $1 > \gamma > \delta$, at least $n - 1$ of the eigenvalues of **A** are in an open disc of radius γ around the origin.

Given the specification of the mean in the theorem, *dom***A** tends to one and **A** tends to be stochastic (i.e. **A** tends to fulfil $\mathbf{eA} = \mathbf{e}$). It turns out that the subdominant eigenvalues tend to zero not only for random matrices with a common mean for all elements of the matrix, but it suffices—given the other assumptions—that each row has its own mean. Intuitive argument: if the rows of **A** have mean c_i/n, $\bar{\mathbf{A}} = (a_{ij} / c_i)$ has mean $1/n$. Note that we should *reduce* the generality of our analysis if we postulated that both the rows *and* the columns of the input matrix were i.i.d. (cf. Section 5).